Assignment 2 - Solutions : Total : 25 points

① Let $P \equiv ((p \to q) \to (r \to q)) \to (p \to r)$.

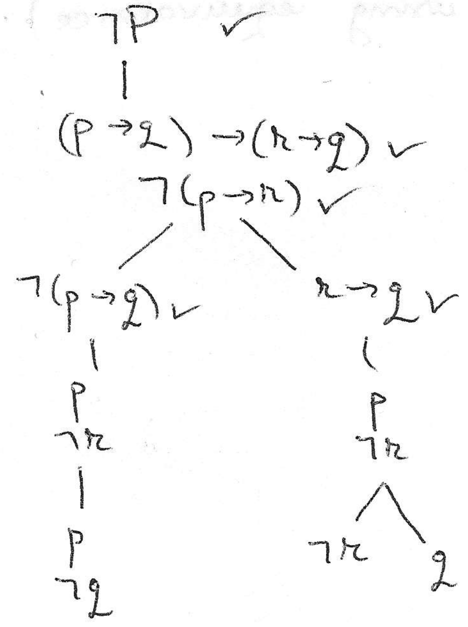
4 points To determine if P is a tautology, we construct a truth tree for $\neg P$. Any complete active paths will give counterexamples.

We write: $\neg P \equiv \neg\Big(\big((p \to q) \to (r \to q)\big) \to (p \to r)\Big) \equiv$

$\equiv \neg\big(\neg((p \to q) \to (r \to q)) \vee (p \to r)\big)$

$\equiv \big((p \to q) \to (r \to q)\big) \wedge \neg(p \to r)$.

3p →



There are 3 complete active paths. We conclude that $\neg P$ is T when $p = T$, $q = F$, $r = F$ or $p = T$, $q = T$, $r = F$. Hence, P is not a tautology.

Counterexamples:

1p →

$p = T,\ q = F,\ r = F$

$p = T,\ q = T,\ r = F$

(2) (4 points)

$H_1: x \to (y \vee z)$

$H_2: y \to w$

$H_3: \neg z$

$C \therefore x \to w$

(1) $\neg x \vee (y \vee z)$ (from H_1, using equivalence)

(2) $\neg y \vee w$ (from H_2, using equivalence)

(3) $(\neg x \vee y) \vee z$ (from (1), using associativity)

(4) $\neg x \vee y$ (from (3) and H_3, using disjunctive syllogism)

(5) $\neg x \vee w$ (from (2) and (4), using resolution)

(C) $x \to w$ (from (5), using equivalence)

(9 points)

(3) (a) The translation is:

$m \to a$

$(\neg a \wedge \neg d) \to \neg b$

1p → $b \vee \neg m$

$\therefore d \to b$

(4 points)

(b) (i) Truth table ← 3p

a	m	d	b	$m \to a$	$(\neg a \wedge \neg d) \to \neg b$	$b \vee \neg m$	$d \to b$	
T	T	T	T	T	T	T	T	←
T	T	T	F	T	T	F		
T	T	F	T	T	T	T	T	←
T	T	F	F	T	T	F		
T	F	T	T	T	T	T	T	←
T	F	T	F	T	T	T	F	← !
T	F	F	T	T	T	T	T	←
T	F	F	F	T	T	T	T	←
F	T	T	T	F	T	T		
F	T	T	F	F	T	F		

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a	m	d	b	$m \to a$	$(\neg a \wedge \neg d) \to \neg b$	$b \vee \neg m$	$d \to b$	
F	T	F	T	F	F	T		
F	T	F	F	F	T	F		
F	F	T	T	T	T	T	T	←
F	F	T	F	T	T	T	F	← !
F	F	F	T	T	F	T		
F	F	F	F	T	T	T	T	←

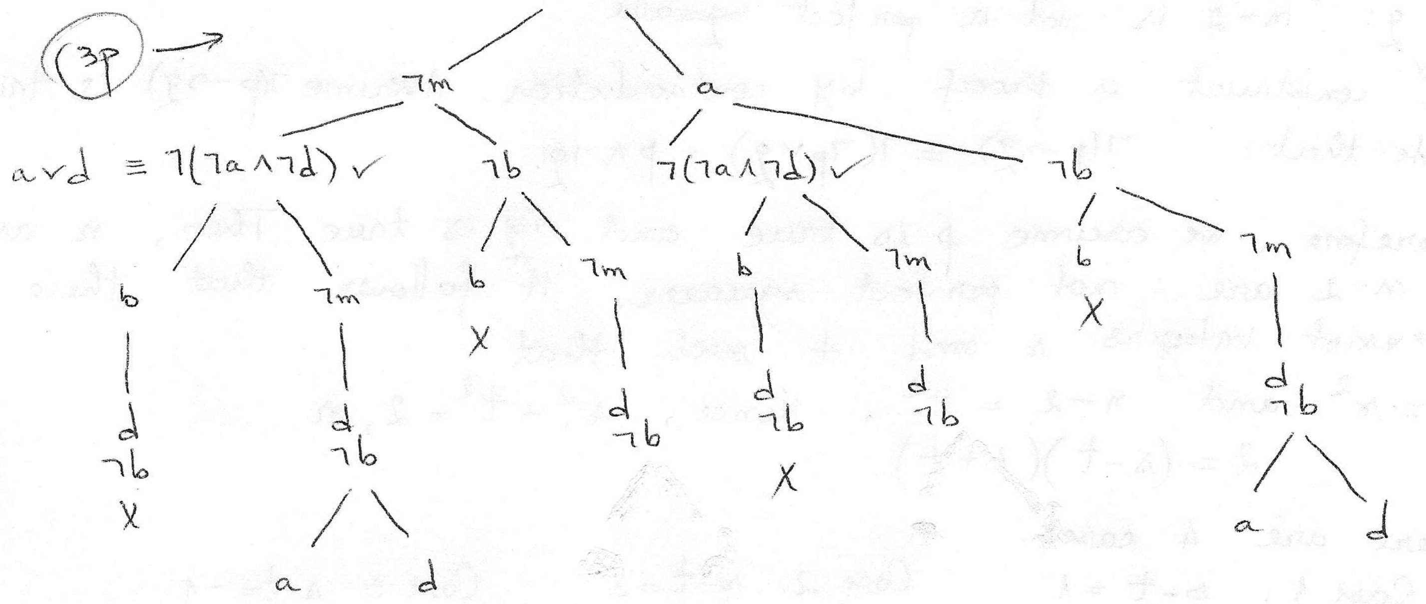
Note: The sign ← marks the rows where all premises are true
! marks the rows where all premises are true, but the conclusion is false

The argument is invalid. Counterexamples: $a = T, m = F, d = T, b = F$
(1p) → $a = F, m = F, d = T, b = F$

(ii) Truth tree. (4points)

$m \to a$ ✓
$(\neg a \wedge \neg d) \to \neg b$ ✓
$b \vee \neg m$ ✓
$\neg(d \to b)$ ✓

(3p) →



There exist complete active paths in the tree so the argument is invalid.
Counterexamples: $a = T, m = F, d = T, b = F$
(1p) → $a = F, m = F, d = T, b = F$

(4) The statement is of the form $p \to q$, where:

(4 points) p: " a and b are two rational numbers".

q: " $a^2 b + 3$ is a rational number ".

For a direct proof, assume p is true and prove ~~but~~ that q is true.

Proof: Assume a and b are two rational numbers. Then, there exist $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ with $n \neq 0$ and $s \in \mathbb{Z}$ and $t \in \mathbb{Z}$, $t \neq 0$

(1p) such that:

$$a = \frac{m}{n} \quad \text{and} \quad b = \frac{s}{t}.$$

We have: $a^2 \cdot b + 3 = \frac{m^2}{n^2} \cdot \frac{s}{t} + 3 = \frac{m^2 s + 3n^2 t}{n^2 \cdot t} = \frac{k}{l}$,

(2p)

where $k = m^2 s + 3n^2 t \in \mathbb{Z}$, since $m, n, s, t \in \mathbb{Z}$

$l = n^2 t \in \mathbb{Z}$ and $l \neq 0$, since $n, t \in \mathbb{Z}$ and $n \neq 0$, $t \neq 0$

(1p)

Hence, $a^2 b + 3 \in \mathbb{Q}$.

(5) The statement is of the form $p \to q$, where

(4 points) p: " n is a perfect square"

q: " n−2 is not a perfect square".

(1p) We construct a proof by contradiction. Assume $\neg(p \to q)$ is true.

Note that: $\neg(p \to q) \equiv \neg(\neg p \vee q) = p \wedge \neg q$.

Therefore, we assume p is true and $\neg q$ is true. Then, n and n−2 are not perfect squares. It follows that there

(1p) exist integers s and t such that

$n = s^2$ and $n - 2 = t^2$. Hence, $s^2 - t^2 = 2$, or

$2 = (s-t)(s+t)$.

There are 4 cases.

Case 1: $s - t = 1$, $s + t = 2$; $2s = 3 \Rightarrow s = \frac{3}{2} \notin \mathbb{Z}$

Case 2: $s - t = 2$, $s + t = 1$; $2s = 3 \Rightarrow s = \frac{3}{2} \notin \mathbb{Z}$

Case 3: $s - t = -1$, $s + t = -2$; $2s = -3 \Rightarrow s = -\frac{3}{2} \notin \mathbb{Z}$

(2p)

Case 4: $s - t = -2$, $s + t = -1$; $2s = -3 \Rightarrow s = -\frac{3}{2} \notin \mathbb{Z}$

Each case gives a contradiction, so we conclude that $p \to q$ is true